

SMALL-SCALE FLOWS AND SURFACE EFFECTS IN THE HYDROMECHANICS OF MULTIPHASE MEDIA

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On the basis of concepts about multiphase continuous media expounded in [1], equations for mechanics of a two-phase dispersed mixture are obtained within the framework of a "two-pressure", two-velocity and two-temperature model with consideration of small-scale motion near the inclusions, of phase transitions, of surface energy, and of compressibility of both components. The small-scale motion is here understood to be the motion of the carrier medium near the inclusions due to their radial pulsation and relative motion in the carrier phase. The introduction of a surface phase made it possible to utilize equations of heat influx for each phase in the form which leads to a more convenient method for taking into consideration effects related to phase transformations. The basic concepts of thermodynamics for such media are examined. An explicit expression is obtained for production of entropy of the mixture. Proceeding from this expression it is possible to formulate linear phenomenological relationships which characterize the phase interactions. This procedure is carried out for the example of the equation of kinetics of phase transitions.

As an area of application where it is necessary to take into account the indicated effects simultaneously, it is possible to point out the flow of fluids containing bubbles of gas (or vapor). A particular application is the propagation of perturbations in such mixtures. When pressures and temperatures are sufficiently close to the critical, it may be necessary to take into account the compressibility of both the vapor and the liquid.

In more special cases the hydromechanical equations of fluid motion with gas bubbles were investigated in papers [2-4] within the framework of a "two-pressure" model (taking into account radial small-scale motion around the bubble). Papers [5-8] are devoted to the propagation of small disturbances and shock waves in such media. Paper [6] contains an interesting analysis of fundamental physical phenomena in a fluid containing gas bubbles.

1. Let us examine the heterogeneous mixture of two compressible phases or substances in each of which there are no effects due to rigidity. The second phase is present in the form of individual spherical inclusions (bubbles, particles, drops) of equal radius a . The direct interaction (for example collisions) between inclusions can be neglected. The first phase is the carrier medium. It is assumed that the fundamental assumption of mechanics of continuous media is valid, i. e. that the distances over which the flow parameters of the mixture change substantially (outside of the surface of discontinuity) are much greater than the characteristic dimensions of the mentioned inclusions and of distances between them. This allows us to describe the dispersed mixture as the sum of two (or more, if the dimensions of inclusions can be represented in the form of a discrete set of values) continua which occupy the same volume. In each point of the volume which

is occupied by the mixture it is possible to introduce macroscopic velocities of phases \mathbf{v}_i , pressures p_i , volumetric content of phases α_i , average densities ρ_i , etc.

$$\rho = \rho_1 + \rho_2, \quad \alpha_1 + \alpha_2 = 1, \quad \alpha_i \geq 0, \quad \rho_i = \rho_i^0 \alpha_i, \quad \alpha_2 = 4/3 \pi a^3 n \quad (1.1)$$

Here ρ is the density of the mixture, ρ_i^0 is the true density of the substance of the i th phase (of the i th component), n is the number of suspended particles (bubbles) in the unit volume of the mixture. Following paper [1], the differential equations for mass and momentum of each phase, and also an equation for the number of particles, are represented in the form

$$\begin{aligned} \frac{\partial \rho_i}{\partial t} + \nabla (\rho_i \mathbf{v}_i) &= J_{ji} - J_{ij}, & \frac{\partial n}{\partial t} + \nabla (n \mathbf{v}_2) &= \psi \\ \rho_i \frac{d_i \mathbf{v}_i}{dt} &= \nabla^k \sigma_i^k + \mathbf{r}_{ji} + J_{ji} (\mathbf{v}_{ji} - \mathbf{v}_i) - J_{ij} (\mathbf{v}_{ij} - \mathbf{v}_i) + \rho_i \mathbf{F}_i \\ \left(\frac{d_i}{dt} = \frac{\partial}{\partial t} + \mathbf{v}_i \nabla = \frac{\partial}{\partial t} + v_i^k \nabla^k = \frac{\partial}{\partial t} + v_i^k \frac{\partial}{\partial x^k}; \mathbf{r}_{ji} = -\mathbf{r}_{ij}, \quad i, j = 1, 2; i \neq j \right) \end{aligned} \quad (1.2)$$

Here and in the following text, summation is only over the superscripts relative to coordinate axes. Further, J_{ij} are "observable" macroscopic rates of phase transformation ($J_{ij} \geq 0$), which only give the rate of formation of the j th phase (in the opposite case $J_{ij} = 0$) at the expense of the i th phase per unit volume and time. The quantity \mathbf{v}_{ij} characterizes the momentum of the mass which is undergoing the phase transition $i \rightarrow j$; ψ characterizes the change in the number of particles of the dispersed phase due to processes of breaking conglomeration or formation of new inclusions. In this paper the case will be examined where such processes are absent, i. e.

$$\psi = 0 \quad (1.3)$$

Further, \mathbf{r}_{ji} is the interaction force between phases with reference to a unit of mixture volume; \mathbf{F}_i are external mass forces; σ_i is the tensor of surface forces acting on the i th phase.

For dispersed systems where the volumetric content of the suspended phase α_2 is sufficiently small, we can assume [1]

$$\sigma_1^{kl} = -p_1 \delta^{kl} + \tau_1^{kl}, \quad \sigma_2^{kl} = 0 \quad (1.4)$$

here δ^{kl} is the Kronecker symbol, p_1 and τ_1^{kl} are the pressure and the tensor of viscous stresses of the carrier medium. The latter is determined by the external rate of deformation tensor

$$e_1^{kl} = \frac{1}{2} \left(\frac{\partial v_1^k}{\partial x^l} + \frac{\partial v_1^l}{\partial x^k} \right) \quad (1.5)$$

The effect of displacements on interphase boundaries can be taken into account through corrections in the coefficients of viscosity. The interaction force between phases \mathbf{r}_{12} are represented in the form

$$\mathbf{r}_{12} = \mathbf{f}_j + \mathbf{f}_m + \mathbf{f}_r - \alpha_2 \nabla p_1 = \mathbf{f}_j + \mathbf{r} - \alpha_2 \nabla p_1, \quad \mathbf{r} = \mathbf{f}_m + \mathbf{f}_r \quad (1.6)$$

where \mathbf{f}_j is the friction force (Stokes force) between phases due to viscous forces, \mathbf{f}_m is the force of "associated" masses, \mathbf{f}_r is the force due to gradients in the velocity field of the carrier phase \mathbf{v}_1 (Magnus or Zhukovskii force), the last term $\alpha_2 \nabla p_1$ is connected with the influence of the pressure field of the carrier phase on the inclusions (Archimedes force). For these forces the following relationships can be utilized:

$$\begin{aligned} \mathbf{f}_f &= \alpha_3 \rho_1 \circ K (\mathbf{v}_1 - \mathbf{v}_2) & (K = K(\alpha_2, |\mathbf{v}_1 - \mathbf{v}_2|, a, \dots)) \\ \mathbf{f}_m &= \alpha_3 \rho_1 \circ \chi' \frac{d\alpha_2}{dt} (\mathbf{v}_1 - \mathbf{v}_2), & \mathbf{f}_r = \alpha_4 \rho_1 \circ \chi'' (\mathbf{v}_1 - \mathbf{v}_2) \times \text{rot } \mathbf{v}_1 \end{aligned} \quad (1.7)$$

Coefficients K , χ' and χ'' take into account the effect due to the shape of particles, due to their nonsingle behavior, and other factors. The elimination of the term \mathbf{r} in (1.6) is necessary for later development and is explained by the fact that \mathbf{r} is not connected directly with effects of viscosity, just as the force of Archimedes.

2. For later use it is necessary to improve the accuracy of the expression for the energy per unit mass of the mixture E , which is composed of the internal energy u and kinetic energy k of the medium

$$E = u + k \quad (2.1)$$

In theories of interpenetrating flow (see paper [1] and appropriate references there) the energy of each component is assumed to be proportional to its mass, and the energy of the mixture is considered additive according to the mass of the phases

$$\rho u = \rho_1 u_1 + \rho_2 u_2, \quad \rho k = \frac{1}{2} \rho_1 v_1^2 + \frac{1}{2} \rho_2 v_2^2 \quad (2.2)$$

Such a representation (first equation (2.2)) ignores peculiarities if the surface layer with a thickness of the order of the radius of molecular interaction in the condensed phase (appr. 10^{-9} m) which represents the interface of the phases. This so-called surface or capillary effect is taken into account (according to Gibbs) by introducing [9, 10] a surface component of internal energy of mixture (and correspondingly some additional surface or capillary phase, which sometimes is referred to as the σ -phase) proportional to the interface of phases

$$\rho u = \rho_1 u_1 + \rho_2 u_2 + n u_\sigma^* \quad (u_\sigma^* = 4\pi a^2 u_\sigma) \quad (2.3)$$

Here u_σ^* and u_σ are surface energies for one inclusion and per unit surface, respectively. The latter, just as the internal specific energy of phases u_i , is a thermodynamic parameter. Further, we take advantage of the hypothesis of local equilibrium between phases and the capillary layer (this allows to introduce the temperatures T_i ($i = 1, 2$) and T_σ), respectively) and the following equations of state:

$$\begin{aligned} u_i &= u_i(\rho_i, T_i), & p_i &= p_i(\rho_i, T_i), & s_i &= s_i(\rho_i, T_i) \\ u_\sigma &= \sigma - T_\sigma \frac{d\sigma}{dT_\sigma}, & s_\sigma &= - \frac{d\sigma}{dT_\sigma}, & \sigma &= \sigma(T_\sigma) \end{aligned} \quad (2.4)$$

where s_i and s_σ are the respective specific entropies of the phases and the unit surface of their interface and σ is the coefficient of surface tension which depends on temperature and the material of both phases. The introduction of the concept of the σ -phase can be related to not only the capillary effect but also to the consideration of special properties of thin layers around inclusions of a different character (for example, a flame around a burning particle when this flame has a temperature substantially different from the temperature of both the particle and the carrier medium).

Equations of state (2.4) obey the Gibbs relationships

$$T_i \frac{d_i s_i}{dt} = \frac{d_i u_i}{dt} + p_i \frac{d_i}{dt} \frac{1}{\rho_i^\circ}, \quad T_\sigma \frac{d_2 s_\sigma^*}{dt} = \frac{d_2 u_\sigma^*}{dt} - \sigma \frac{d_2}{dt} (4\pi a^2) \quad (2.5)$$

The kinetic energy of the mixture is presented in (2.2) only through the energy of the macroscopic motion of the phases with velocities \mathbf{v}_i . In some cases (especially in the

case of motion of a fluid with bubbles, or in general, when the density of the material of the carrier phase ρ_1° is greater than or of the same order as the density of the material of the suspended phase ρ_2° it is necessary also to take into account the kinetic energy of small-scale flows (with characteristic linear dimensions equal to the order of the dimensions of inclusions), the macroscopic momentum of which is equal to zero. This in the first place is the energy of radial pulsating motion around the bubble k_p and the energy of perturbations k_v , which will be called pseudo-rotational, in the carrier phase due to relative motion of inclusions within the phase (k_p and k_v are the corresponding energies per one inclusion). In this manner we have

$$\rho k = {}^{1/2}\rho_1 v_1^2 + {}^{1/2}\rho_2 v_2^2 + n k_p + n k_v \quad (2.6)$$

Relationships (2.3) and (2.6) represent the general additivity of internal and kinetic energy of the mixture. The other thermodynamic potentials of the mixture (entropy, free energy, and others) in this case are also determined in a manner analogous to (2.3).

We select a fixed volume V , bounded by a stationary surface S . In analogy to [1] we define the concept of substantive derivative of the total energy of the mixture. This derivative represents the change of energy in a stationary volume which is due only to the interaction with the external medium and is not connected with the convective transport of material, i. e. we have

$$\int_V \frac{\partial \rho E}{\partial t} dV = - \int_S \left[\rho_1 \left(u_1 + \frac{v_1^2}{2} \right) v_{1n} + \rho_2 \left(u_2 + \frac{v_2^2}{2} \right) v_{2n} + (u_\sigma^* + k_p + k_v) \times \right. \\ \left. \times v_{2n} \right] dS + \int_V \rho \frac{dE}{dt} dV \quad (2.7)$$

Applying the Gauss-Ostrogradskii formula, the following expression is obtained for the substantive derivative of the total energy of the medium:

$$\rho \frac{dE}{dt} = \rho_1 \frac{d_1}{dt} \left(u_1 + \frac{v_1^2}{2} \right) + \rho_2 \frac{d_2}{dt} \left(u_2 + \frac{v_2^2}{2} \right) + \\ + n \frac{d_\sigma}{dt} (u_\sigma^* + k_p + k_v) + (J_{12} - J_{21}) (u_2 - u_1 + {}^{1/2}(v_2^2 - v_1^2)) \quad (2.8)$$

For other, in the generalized sense, additive functions which apply to the whole mixture, it is also easy to determine the substantive derivatives analogous to (2.8).

3. Taking into account (1.6), the equation for kinetic energy of the macroscopic motion of the medium follows from (1.2)

$$\rho_1 \frac{d_1}{dt} \frac{v_1^2}{2} + \rho_2 \frac{d_2}{dt} \frac{v_2^2}{2} = v_1 \nabla^k \sigma_1^k + v_2 \nabla^k \sigma_2^k - (J_{12} - J_{21}) \frac{v_2^2 - v_1^2}{2} + \rho_1 \mathbf{F}_1 v_1 + \\ + \rho_2 \mathbf{F}_2 v_2 - \mathbf{r}_{12} (v_1 - v_2) + J_{21} (v_1 - v_2) \left(v_{21} - \frac{v_1 + v_2}{2} \right) - J_{12} (v_1 - v_2) \times \\ \times \left(v_{12} - \frac{v_1 + v_2}{2} \right) \quad (3.1)$$

An analysis of (3.1), (1.2) and the motion of bodies in the fluid shows that the force interaction between the phases leads to the following: (1) a transfer of kinetic energy of macroscopic motion from one phase to the other; (2) dissipation of energy of macroscopic motion into internal energy of the mixture (into heat) because of viscous forces with the intensity $\mathbf{f}_r (v_1 - v_2)$; (3) an exchange between the energy of macroscopic motion and the energy of small-scale pseudo-rotational flows $n k_v$, with an intensity

$\mathbf{r} (\mathbf{v}_1 - \mathbf{v}_2)$; due to the action of components r_{12} which have a "nonviscous" nature (Magnus force, effect of associated masses); (4) an exchange of energy with the external medium due to the pressure gradient ∇P_1 , i. e. at the expense of surface forces.

The absence of an energy concept for small-scale flows in the theories of interpenetrating motion makes it necessary to relate the effect noted in point 3, to pure dissipation, i. e. to a transfer of kinetic energy directly into heat. We note that in the case of flow around the inclusions with sufficiently high velocities, additional terms can arise in the force \mathbf{r} . These terms transform the kinetic energy of macroscopic motion into rotational motion in the trail of particles.

The kinetic energy equation of small-scale motion nk_v , which arises due to differences in phase velocities, can be written in the form

$$n \frac{dk_v}{dt} = \mathbf{r} (\mathbf{v}_1 - \mathbf{v}_2) - n\eta_{v1} - n\eta_{v2} \tag{3.2}$$

As was mentioned previously, here the first term in the right side determines the exchange with the kinetic energy of macroscopic motion due to force interaction of phases. The second and the third term $n\eta_{vi}$ ($i = 1, 2$) determine the dissipation of energy of small-scale pseudo-rotational motion into internal (thermal) energy of the first and the second phase, respectively.

As an approximation it is possible to neglect the motion (and dissipation) within the inclusions and to determine η_{v1} as a dissipation function [11]

$$\eta_{v1} = \frac{\rho_1^\circ v_1}{2} \int_{R>0} \left(\frac{\partial w^i}{\partial x^j} + \frac{\partial w^j}{\partial x^i} \right) \left(\frac{\partial w^i}{\partial x^j} + \frac{\partial w^j}{\partial x^i} \right) dV = - \rho_1^\circ v_1 \int_{R=a} \frac{\partial}{\partial n} w^2 dS \tag{3.3}$$

(v_1 is the kinematic viscosity coefficient of the first phase). This function can be determined by the velocity field $w^i(x^j)$ which is generated by the motion of a sphere with the velocity w_v in an ideal incompressible fluid. The potential φ_v and the kinetic energy of perturbation k_v can be used for the description of the function

$$\varphi_v = \frac{w_v a^3}{2} \frac{x}{R^3}, \quad k_v = \frac{2\pi}{3} a^3 \rho_1^\circ w_v^2 \tag{3.4}$$

Taking into consideration (3.3) and (3.4), we have

$$\eta_{v1} = 36\pi a v_1 \rho_1^\circ w_v^2 = 54 v_1 a^{-2} k_v, \quad \eta_{v2} = 0 \tag{3.5}$$

4. Let us examine the problem of bubble pulsation in an incompressible fluid with density ρ_1° using the Rayleigh formulation [11] and taking into account phase transitions on the interface. This motion is described by the potential of the velocity field φ , the Cauchy-Lagrange integral

$$\varphi = - \frac{a^2 w_a^2}{R}, \quad \frac{\partial \varphi}{\partial t} + \frac{w^2}{2} + \frac{p}{\rho_1^\circ} = F(t) \tag{4.1}$$

and the boundary conditions

$$\begin{aligned} R = a, & \quad w = w_a, & \quad p = p_a = p_0 - 2\sigma/a \\ R = \infty, & \quad w = 0, & \quad p = p_\infty \end{aligned} \tag{4.2}$$

Here p_0 is the pressure in the bubble. Furthermore, the following relationship between the velocity a' of the interface and the mass velocity of the fluid on this surface w_a is valid

$$a' = w_a + (j_{12} - j_{21})/\rho_1^\circ \tag{4.3}$$

Here j_{ij} is the velocity of phase transformation per unit of surface. The kinetic energy

k of the motion which is under consideration and the derivative of this motion with respect to time k' have the form

$$k = 4\pi\rho_1 \int_a^\infty \frac{w^2}{2} R^2 dR = 2\pi\rho_1 a^3 w_a^2, \quad k' = 4\pi\rho_1 a^2 w_a \left(\frac{3}{2} a' w_a + a w_a' \right) \quad (4.4)$$

It follows from (4.1)–(4.4) that

$$k' = 4\pi a^2 [(p_0 - 2\sigma/a - p_\infty) w_a + 1/2 (j_{21} - j_{12}) w_a^2] \quad (4.5)$$

The effect of viscosity can be taken into account in a manner which is analogous to (3.3). Into the right side of (4.5) we introduce a dissipation function which is determined by the velocity field (4.1)

$$\eta_{p1} = 16\pi\rho_1 \nu_1 a w_a^2 = 8\nu_1 k a^{-2} \quad (4.6)$$

In this manner we obtain a kinetic energy equation

$$k' = 4\pi a^2 [(p_0 - 2\sigma/a - p_\infty) w_a + 1/2 (j_{21} - j_{12}) w_a^2] - 8\nu_1 k a^{-2} \quad (4.7)$$

which corresponds to the generalized equation of Lamb for the pulsation of a small bubble in an incompressible fluid, taking into account surface tension, phase transitions, and viscous dissipation

$$\frac{3}{2} w_a^2 + a w_a' + \frac{2(j_{12} - j_{21})}{\rho_1} w_a = \frac{p_0 - p_\infty - 2\sigma/a}{\rho_1} - \frac{4\nu_1}{a} w_a \quad (4.8)$$

The relationship for the energy change of small-scale pulsating radial motion in a two-phase system will be specified. We shall proceed from Eqs. (4.7) or (4.8). We will consider them as kinetic relationships and make the following identifications: the pressure distant from the bubble (p_∞) and the pressure inside the bubble (p_0) are identified with the pressure of the first (p_1) and second (p_2) phase, respectively; the quantity k is identified with the kinetic energy of the pulsating radial motion k_p for one inclusion; the velocity w_a is identified with the velocity w_{p1} which determines the radial velocity of the carrier medium on the interface of phases. As a result we have

$$a \frac{d_2 w_{p1}}{dt} + \frac{3}{2} w_{p1}^2 + \frac{J_{12} - J_{21}}{2\pi a^2 n \rho_1} w_{p1} = \frac{p_2 - p_1 - 2\sigma/a}{\rho_1} - \frac{4\nu_1}{a} w_{p1} \\ \frac{d_2 a}{dt} = w_{p1} + \frac{J_{12} - J_{21}}{4\pi n \rho_1 a^2} \quad (4.9)$$

The equation for the energy of small-scale radial motion corresponding to Eq. (4.9) is

$$n \frac{d_2 k_p}{dt} = 4\pi a^2 n (p_2 - p_1 - 2\sigma/a) w_{p1} - n \eta_{p2} - n \eta_{p1} \quad (4.10)$$

$$k_p = 2\pi\rho_1 a^3 w_{p1}^2, \quad \eta_{p1} = 8\nu_1 k_p a^{-2}, \quad \eta_{p2} = 1/2 (J_{12} - J_{21}) w_{p1}^2 n^{-1}$$

The first term in the right side of (4.10) describes the transition of kinetic energy into energy of compression and surface tension and the reverse. The second term describes the inflow (due to the transition $2 \rightarrow 1$) or the sink (due to the transition $1 \rightarrow 2$) of kinetic energy at the expense of the internal energy of the second phase. Finally, the third term describes the dissipation of kinetic energy into internal (thermal) energy of the first phase.

5. For further determination of the behavior of the system it is necessary to utilize the equations for inflow of heat of each phase [12]. In the general case these equations can

be represented in the form

$$\begin{aligned} \rho_i \frac{d_i u_i}{dt} &= \rho_i A_i - J_{ji} x'_{i,j} - J_{ij} x_{i,j} - q_{i\sigma} - \nabla \mathbf{q}_i + \rho_i Q_i \quad (i, j = 1, 2; i \neq j) \\ n \frac{d_2 u_\sigma}{dt} &= n A_\sigma - J_{12} x_{\sigma,12} - J_{21} x_{\sigma,21} + q_{1\sigma} + q_{2\sigma} \end{aligned} \quad (5.1)$$

Here A_i ($i = 1, 2, \sigma$) represents the power of internal forces per unit mass of the phase or per one inclusion, respectively. The remaining terms represent the inflow of heat; $x_{l,j}$ ($l = 1, 2, \sigma; i, j = 1, 2; i \neq j$) is the heat flow from the l th phase to the material which has undergone the transformation $j \rightarrow i$ with reference to the mass of the material; $q_{i\sigma} = -q_{\sigma i}$ is the heat flow per unit volume of mixture from the i th phase to the interface unrelated to phase transitions; \mathbf{q}_i and $\rho_i Q_i$ are the external surface heat flow (thermal conductivity and radiation) and the power of volume heat sources, respectively.

For dispersed systems which we are studying it usually can be assumed that

$$\mathbf{q}_1 = -\lambda_1 \nabla T_1, \quad q_2 = 0, \quad q_{i\sigma} = 4\pi a^2 n \beta_i (T_i - T_\sigma) \quad (5.2)$$

where λ_i is the coefficient of thermal conductivity of the material in the i th phase, β_i is the coefficient of heat transfer from the interface to the i th phase. For this coefficient we can use experimental relationships of the form

$$\begin{aligned} N_{Nu, i} &= F_i (N_{Pr, i}, N_{Re}) \\ \left(N_{Nu, i} = \frac{2\beta_i a}{\lambda_i}, \quad N_{Pr, i} = \frac{c_{pi} v_i \rho_i}{\lambda_i}, \quad N_{Re} = \frac{2|v_1 - v_2| a}{v_1} \right) \end{aligned} \quad (5.3)$$

where c_{pi} is the heat capacity at constant pressure of the i th phase.

In general the determination of the so-called work of internal forces of each component is connected with the description of the behavior of each inclusion. This becomes an insurmountable (and for the description in the average an unnecessary) task under conditions which are studied here where the number of such inclusions is large. Therefore, following [1], the power of internal forces of the phase are given as an average quantity in terms of the macroscopic parameters proceeding from the analysis of the motion of the body in the fluid and the developed relationships (3.1), (3.2) and (4.10)

$$\begin{aligned} \rho_i A_i &= \frac{\alpha_i p_i}{\rho_i^\sigma} \frac{d_i \rho_i}{dt} + \tau_i^{kl} e_i^{kl} + \kappa_i f_f (v_1 - v_2) + n \eta_{pi} + n \eta_{vi} + J_{ji} \frac{(v_{ji} - v_i)^2}{2} - \\ &\quad - J_{ij} \frac{(v_{ij} - v_i)^2}{2} \\ (\kappa_1 + \kappa_2 = 1; i, j = 1, 2; i \neq j) \end{aligned} \quad (5.4)$$

Here the first term is due to the work of internal pressure forces and compressing the i th phase (reversible work). The remaining terms represent the energy dissipation (uncompensated heat) in the i th phase as a result of internal viscous forces which arise due to gradients in the average velocity field ($\tau_1^{kl} e_1^{kl}$, where e_1^{kl} is the deformation tensor determined by the velocity field v_1), due to viscous interaction with the other phase ($\kappa_i f_f (v_1 - v_2)$), due to shear deformation as a result of radial and pseudo-rotational small-scale flow ($n \eta_{pi}$ and $n \eta_{vi}$), and finally due to irreversible momentum transfer in phase transformations [1] (last two terms). The coefficients κ_i indicate the share of dissipation due to Stokes force f_f , of kinetic energy of the macroscopic motion of the

mixture. This share is transferred directly into the internal energy of the i th phase. In many cases where we can neglect the shear deformations of the inclusions with respect to the shear deformations of the carrier medium, the following relationships are valid:

$$\alpha_1 = 1, \quad \alpha_2 = 0, \quad v_{21} = v_{12} = v_2 \quad (5.5)$$

The power due to internal forces of the σ -phase can be represented in the following form [9, 10]

$$A_\sigma = 4\pi a^2 \frac{2\sigma}{a} \frac{d_2 a}{dt} = 8\pi a \sigma \frac{d_2 a}{dt} \quad (5.6)$$

This corresponds to the work of internal forces in a spherical uniformly stretched elastic film in which the tension per unit length is equal to σ .

Proceeding from definition (2, 8) and the equations for the energy components of the mixture (3, 1), (3, 2), (4, 10), (5, 1) and taking into account (1, 4), (1, 6), we obtain the explicit expression for the substantive derivative of the total energy of the mixture

$$\begin{aligned} \rho \frac{dE}{dt} = & J_{12} \left[u_2 - u_1 - (x_{1,12} + x_{2,12} + x_{\sigma,12}) + p_2 \left(\frac{1}{\rho_2^0} - \frac{1}{\rho_1^0} \right) + \frac{2\sigma}{a} \frac{1}{\rho_1^0} \right] - \\ & - J_{21} \left[u_2 - u_1 + (x_{1,21} + x_{2,21} + x_{\sigma,21}) + p_2 \left(\frac{1}{\rho_2^0} - \frac{1}{\rho_1^0} \right) + \frac{2\sigma}{a} \frac{1}{\rho_1^0} \right] - \\ & - \nabla p_1 (\alpha_1 v_1 + \alpha_2 v_2) + \nabla^k \tau_1^{kl} v_1^l - \nabla q_1 + \rho_1 F_1 v_1 + \rho_2 F_2 v_2 + \rho_1 Q_1 + \rho_2 Q_2 \end{aligned} \quad (5.7)$$

The total energy of the medium can change only as a result of external influences (described by the last eight terms), but in no way as a result of internal processes. Therefore, the expressions in brackets in the right side of (5, 7) which have the nature of sources connected with phase transitions, must be equal to zero, i. e.

$$\begin{aligned} x_{1,12} + x_{2,12} + x_{\sigma,12} &= u_2 - u_1 + p_2 \left(\frac{1}{\rho_2^0} - \frac{1}{\rho_1^0} \right) + \frac{2\sigma}{a\rho_1^0} = i_2 - i_1 - \frac{p_2 - p_1 - 2\sigma/a}{\rho_1^0} \\ x_{1,21} + x_{2,21} + x_{\sigma,21} &= u_1 - u_2 + p_2 \left(\frac{1}{\rho_1^0} - \frac{1}{\rho_2^0} \right) - \frac{2\sigma}{a\rho_1^0} = i_1 - i_2 + \frac{p_2 - p_1 - 2\sigma/a}{\rho_1^0} \\ (i_j &= u_j + p_j/\rho_j^0, \quad j = 1, 2) \end{aligned} \quad (5.8)$$

Here i_j is the enthalpy of the corresponding phase.

In order to obtain a concrete model, it is necessary to determine the energies $x_{l,ij}$ which determine the energy loss of the l th phase in the transformation $i \rightarrow j$ of a unit mass for the case of known total (with respect to l) effects (5, 8). These supplementary relationships (analogous to the functions for accommodation coefficients in the kinetic theory of gasses where these functions reflect the interaction of the medium with surfaces) must be postulated.

As a possible variation of this type of relationship, some generalization of functions [1] which were applied to the two-phase medium with equal pressures of phases ($p_1 = p_2$) and in the absence of capillary effects can be used. That is, the relationships for $x_{l,ij}$ which in some analogy to the kinetic theory of gasses can be called accommodational, have the form:

$$\begin{aligned} x_{1,12} &= i_{2s}(p_2) - i_1, & x_{2,12} &= i_2 - i_{2s}(p_2) - \frac{p_2 - p_1}{\rho_1^0}, & x_{\sigma,12} &= \frac{2\sigma}{a\rho_1^0} \\ x_{2,21} &= i_{1s}(p_2) - i_2, & x_{1,21} &= i_1 - i_{1s}(p_2) + \frac{p_2 - p_1}{\rho_1^0}, & x_{\sigma,21} &= -\frac{2\sigma}{a\rho_1^0} \end{aligned} \quad (5.9)$$

Here the subscript s refers to the state of saturation above a planar interface of phases. The utilization of characteristic enthalpies of phases i_{js} in (5, 9) at the saturation condition taken for the pressure p_{2s} is explained by the fact that the phase transitions take

place at the interface, on one side of which the pressure is equal to p_2 and on the other side to $p_2 - 2\sigma/a$.

The most significant feature shown in (5.9) is that the energy which is equal to the heat of transition $i \rightarrow j$, is primarily provided by the i th phase. Another possible and sometimes more convenient variation of accommodation relationships is

$$x_{1,12} = x_{2,12} = x_{1,21} = x_{2,21} = 0, \quad x_{\sigma,12} = -x_{\sigma,21} = i_2 - i_1 - (p_2 - p_1 - 2\sigma/a) / \rho_1^\circ \quad (5.10)$$

Equations (5.10) give the role of a source or sink of heat, which is required in phase transitions, to the surface phase. We note that the presence of the "reverse" coupling of the σ -phase with the first and the second phase due to $q_{\sigma 1}$ and $q_{\sigma 2}$ leads to a situation where the additional heat source or sink in the σ -phase in comparison with (5.9) is compensated by heat flow from the direction of the first and second phase. The actual results obtained through (5.9) and (5.10) are quite close to each other.

In this manner the system of equations which describes the interpenetrating motion of two-phase dispersed mixture taking into account small-scale motion around the inclusions, difference in pressure between phases, compressibility of phases, and phase transitions on existing interfaces, has the form

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \nabla(\rho_1 \mathbf{v}_1) &= J_{21} - J_{12}, & \frac{d_2 a}{dt} &= w_{p1} + \frac{J_{12} - J_{21}}{4\pi a^2 n \rho_1^\circ} \\ \frac{\partial \rho_2}{\partial t} + \nabla(\rho_2 \mathbf{v}_2) &= J_{12} - J_{21}, & \frac{\partial n}{\partial t} + \nabla(n \mathbf{v}_2) &= 0 \\ \rho_1 \frac{d_1 \mathbf{v}_1}{dt} &= -\alpha_1 \nabla p_1 + \nabla^k \tau_1^{kl} - \mathbf{r} - \mathbf{f}_f + J_{21}(\mathbf{v}_{21} - \mathbf{v}_1) - J_{12}(\mathbf{v}_{12} - \mathbf{v}_1) + \rho_1 \mathbf{F}_1 \\ \rho_2 \frac{d_2 \mathbf{v}_2}{dt} &= -\alpha_2 \nabla p_1 + \mathbf{r} + \mathbf{f}_f - J_{21}(\mathbf{v}_{21} - \mathbf{v}_2) + J_{12}(\mathbf{v}_{12} - \mathbf{v}_2) + \rho_2 \mathbf{F}_2 \\ a \frac{d_2 w_{p1}}{dt} + \frac{3w_{p1}^2}{2} + \frac{(J_{12} - J_{21})w_{p1}}{2\pi a^2 n \rho_1^\circ} &= \frac{p_2 - p_1 - 2\sigma/a}{\rho_1^\circ} - \frac{4v_1 w_{p1}}{a} \\ n \frac{d_2 k_v}{dt} &= \mathbf{r}(\mathbf{v}_1 - \mathbf{v}_2) - n\eta_{v1} - n\eta_{v2}, & \mathbf{r} &= \mathbf{f}_m + \mathbf{f}_r \\ \rho_1 \frac{d_1 u_1}{dt} &= \frac{\alpha_1 f_1}{\rho_1^\circ} \frac{d_1 \rho_1^\circ}{dt} + \tau_1^{kl} e_1^{kl} + \kappa_1 \mathbf{f}_f(\mathbf{v}_1 - \mathbf{v}_2) + n\eta_{v1} + 16\pi n \rho_1^\circ v_1 a w_{p1}^2 + \\ &+ J_{21} \frac{(\mathbf{v}_{21} - \mathbf{v}_1)^2}{2} - J_{12} \frac{(\mathbf{v}_{12} - \mathbf{v}_1)^2}{2} - J_{21} x_{1,21} - J_{12} x_{1,12} - q_{1\sigma} - \nabla \mathbf{q}_1 + \rho_1 Q_1 \\ \rho_2 \frac{d_2 u_2}{dt} &= \frac{\alpha_2 f_2}{\rho_2^\circ} \frac{d_2 \rho_2^\circ}{dt} + \kappa_2 \mathbf{f}_f(\mathbf{v}_1 - \mathbf{v}_2) + n\eta_{v2} + \frac{J_{12} - J_{21}}{2} w_{p1}^2 - J_{21} \frac{(\mathbf{v}_{21} - \mathbf{v}_2)^2}{2} + \\ &+ J_{12} \frac{(\mathbf{v}_{12} - \mathbf{v}_2)^2}{2} - J_{21} x_{2,21} - J_{12} x_{2,12} - q_{2\sigma} + \rho_2 Q_2 \\ 4\pi n a^2 \frac{d_2 u_\sigma}{dt} &= q_{1\sigma} + q_{2\sigma} - J_{21} x_{\sigma,21} - J_{12} x_{\sigma,12}, & u_\sigma &= \zeta(T_\sigma) - T_\sigma \frac{d\zeta}{dT_\sigma} \\ p_i &= p_i(\rho_i^\circ, T_i), & u_i &= u_i(\rho_i^\circ, T_i) \quad (i=1, 2) \\ (\alpha_1 + \alpha_2 = 1, \quad \alpha_2 &= 4/3\pi n a^3, \quad \rho_i = \rho_i^\circ \alpha_i \quad (i=1, 2), \quad \kappa_1 + \kappa_2 = 1) \end{aligned} \quad (5.11)$$

Relationships of the type (1.7), (3.5), (5.2), (5.3), (5.5), (5.9) or (5.10) (taking into

account (5.8) or their generalizations together with equations of kinetics of phase transformations (for J_{ij}) with given external interactions (F_i, Q_i) close to system of equations (5.11) in the domain of continuous motion.

It is important to note that if $v_{12} = v_{21}$ (the most likely case) and if the accommodation relationships (5.10) are assumed, then the necessity of introducing "observed" velocities of phase transitions separately for transformations $1 \rightarrow 2$ (J_{12}) and $2 \rightarrow 1$ (J_{21}) is eliminated. In this case all effects of phase transformations in all equations (5.11) can be taken into account in the form of terms with the factor

$$J = J_{12} - J_{21} \quad (5.12)$$

so that effects of two possible directions of phase transitions are taken into account through a change in sign for the velocity of the resulting reaction J .

6. In analogy to (2.3) we have the generalized additivity for the entropy of the mixture

$$\rho s = \rho_1 s_1 + \rho_2 s_2 + n s_\sigma^* \quad (6.1)$$

For this mixture the expression of substantive derivative (2.8) which represents the change of entropy of the mixture in a fixed volume purely due to the interaction with the external medium and the interaction of internal processes, has the form

$$\rho \frac{ds}{dt} = \rho_1 \frac{d_1 s_1}{dt} + \rho_2 \frac{d_2 s_2}{dt} + n \frac{d_\sigma s_\sigma^*}{dt} + (J_{12} - J_{21})(s_2 - s_1) \quad (6.2)$$

Taking advantage of the Gibbs relationships (2.5) which reflect the local equilibrium of phases, and also using the equations for inflow of heat which enter into system (5.11), we have

$$\begin{aligned} \rho \frac{ds}{dt} = & \frac{\rho_1 Q_1}{T_1} + \frac{\rho_2 Q_2}{T_2} - \nabla \left(\frac{q_1}{T_1} \right) + \mathbf{q}_1 \nabla \left(\frac{1}{T_1} \right) + \frac{\tau_1^{kl} e_1^{kl}}{T_1} + \mathbf{f}_f (v_1 - v_2) \left(\frac{\kappa_1}{T_1} + \frac{\kappa_2}{T_2} \right) + \\ & + \frac{n \eta_{p1}}{T_1} + \frac{n \eta_{v1}}{T_1} + \frac{n \eta_{v2}}{T_2} + q_{1\sigma} \left(\frac{1}{T_\sigma} - \frac{1}{T_1} \right) + q_{2\sigma} \left(\frac{1}{T_\sigma} - \frac{1}{T_2} \right) + J_{21} X_{21} + J_{12} X_{12} \\ X_{21} = & s_1 - s_2 - \frac{x_{1,21}}{T_1} - \frac{x_{2,21}}{T_2} - \frac{x_{\sigma,21}}{T_\sigma} + \frac{(v_{21} - v_1)^2}{2T_1} - \frac{(v_{21} - v_2)^2 + w_{p1}^2}{2T_2} \\ X_{12} = & s_2 - s_1 - \frac{x_{1,12}}{T_1} - \frac{x_{2,12}}{T_2} - \frac{x_{\sigma,12}}{T_\sigma} - \frac{(v_{12} - v_1)^2}{2T_1} + \frac{(v_{12} - v_2)^2 + w_{p1}^2}{2T_2} \end{aligned} \quad (6.3)$$

The first three terms here determine the entropy change due to energy exchange with the external medium. The remaining terms (always nonnegative), which represent the products of thermodynamic forces with thermodynamic fluxes [12], determine the dissipation function. This function gives the generation of entropy due to internal irreversible processes within and between phases.

We introduce the notation

$$\theta_i = \frac{T_i - T_s(p_2)}{T_s(p_2)} \quad (i = 1, 2, \sigma), \quad \theta_{s1} = \frac{T_s(p_1) - T_s(p_2)}{T_s(p_2)} \quad (6.4)$$

Here $T_s(p)$ is the saturation temperature for a planar interface. In the case of small deviations of temperatures of phases T_i and $T_s(p_1)$ from $T_s(p_2)$ (i. e. $|\theta_i|, |\theta_{s1}| \ll 1$) and incompressibility of the carrier phase ($\rho_1^\circ = \text{const}$) the linearized equations of state for the constituents have the form

$$\begin{aligned} s_1(p_1, T_1) &= s_{1s}(p_1) + c_{p1}(\theta_1 - \theta_{s1}) \\ i_1(p_1, T_1) &= i_{1s}(p_1) + c_{p1}(\theta_1 - \theta_{s1}) T_s(p_2) \end{aligned} \quad (6.5)$$

$$s_2(p_2, T_2) = s_{1s}(p_1) - c_{p1} \theta_{s1} + l(p_2) / T_s(p_2) + c_{p2} \theta_2$$

$$i_2(p_2, T_2) = i_{1s}(p_1) + l(p_2) + (c_{p2} \theta_2 - c_{p1} \theta_{s1}) T_s(p_2) + (p_2 - p_1) / \rho_1$$

Here $l(p)$ is the heat of equilibrium transition $1 \rightarrow 2$ on the planar interface (in the case of a fluid with bubbles $l > 0$); c_{p1} and c_{p2} are heat capacities of phases at constant pressure.

From (6.3) and (6.5) we obtain expressions, which are linearized with respect to quantities characterizing the nonequilibrium of pressure and temperature, for the thermodynamic forces of transitions $2 \rightarrow 1$ and $1 \rightarrow 2$

$$X_{21} = \frac{1}{T_s(p_2)} \left[\frac{2\sigma}{a\rho_1^\sigma} + x_{1,21}\theta_1 + x_{2,21}\theta_2 + x_{\sigma,21}\theta_\sigma \right] + \varphi_{21}$$

$$X_{12} = \frac{1}{T_s(p_2)} \left[-\frac{2\sigma}{a\rho_1^\sigma} + x_{1,12}\theta_1 + x_{2,12}\theta_2 + x_{\sigma,12}\theta_\sigma \right] + \varphi_{12} \quad (6.6)$$

$$\left(\varphi_{21} = \frac{(v_{21} - v_1)^2}{2T_1} - \frac{(v_{21} - v_2)^2 + w_{p1}^2}{2T_2}, \quad \varphi_{12} = -\frac{(v_{12} - v_1)^2}{2T_1} + \frac{(v_{12} - v_2)^2 + w_{p1}^2}{2T_2} \right)$$

The quantities φ_{21} and φ_{12} take into account the effect of velocity nonequilibrium on phase transitions. In the case of an equilibrium system we have

$$v_1 = v_2 = v_{12} = v_{21} = w_{p1} = 0, \quad T_1 = T_2 = T_\sigma, \quad p_2 = p_1 + 2\sigma/a,$$

$$X_{12} = X_{21} = 0 \quad (6.7)$$

The known fact follows [9] that the equilibrium temperature $T_s^a(p_2)$ of transitions $2 \rightarrow 1$ on a curved surface of radius a is

$$T_s^a(p_2) = T_s(p_2) \left[1 + \frac{2\sigma}{a\rho_1^\sigma l(p_2)} \right] \quad (6.8)$$

For accommodation relationships (5.9) the expressions (6.6) assume the form

$$X_{21} = -\frac{l(p_2)}{T_s(p_2)} \frac{T_2 - T_s^a(p_2)}{T_s(p_2)} + \varphi_{21}, \quad X_{12} = \frac{l(p_2)}{T_s(p_2)} \frac{T_1 - T_s^a(p_2)}{T_s(p_2)} + \varphi_{12} \quad (6.9)$$

If we take into account the obvious inequalities

$$J_{ij} \geq 0, \quad J_{ij} X_{ij} \geq 0 \quad (6.10)$$

we can examine the linear equations of kinetics for the rates of phase transitions

$$J_{ij} = L_{ij} X_{jj} \quad (X_{ij} > 0), \quad J_{ij} = 0 \quad (X_{ij} \leq 0)$$

$$(L_{ij} \geq 0; \quad i, j = 1, 2; \quad i \neq j) \quad (6.11)$$

In the case of accommodation relationships (5.10) and $v_{21} = v_{12}$ the linearized relationships (6.6) for the thermodynamic forces have the form

$$X = X_{12} = -X_{21} = \frac{l(p_2)}{T_s(p_2)} \frac{T_\sigma - T_s^a(p_2)}{T_s(p_2)} + \varphi_{12} \quad (6.12)$$

The linear equation of kinetics (6.11) can be reduced to an expression for the resultant rate of phase transitions

$$J = J_{12} - J_{21} = LX \quad (L > 0) \quad (6.13)$$

In a large class of problems we can neglect capillary effects (quantities $2\sigma/a$, u_σ^*) or at least the heat capacity of the σ -phase (the quantity $4\pi n a^2 d_2 u_\sigma / dt$). Then the

equation for heat inflow of the surface phase transforms into a finite algebraic equation with respect to T_σ , which according to (6.12) influences the rate of phase transformations

$$q_{1\sigma} + q_{2\sigma} - J_{21}x_{\sigma, 21} - J_{12}x_{\sigma, 12} = 0 \quad (6.14)$$

For example in the case (5.10) taking into account (5.2), we have

$$4\pi na^2 [\beta_1 (T_1 - T_\sigma) + \beta_2 (T_2 - T_\sigma)] - J [i_2 - i_1 - (p_2 - p_1 - 2\sigma/a) / \rho_1^\circ] = 0 \quad (6.15)$$

The obtained results can be generalized for the case of a polydispersed system if the dimensions of inclusions are described by some distribution function. A generalization is also possible to the case where the phases represent homogeneous mixtures of several components entering into chemical reactions.

In the equations describing the phase interactions (for example in (1.7), (3.5), (4.10)) we can take into account with some additional complication, the effect of multiple inclusions using the concepts of the regular "cell" model [13]. We can also take into account the effect of compressibility of the carrier phase on the small-scale pulsating motion.

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